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COMPARISON OF THE SOLUTIONS OF LINEAR AND NON-LINEAR POSITIONAL DIFFERENTIAL GAMES OF ENCOUNTER*

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A positional differential game of encounter of a non-linear, conflict-controlled system with a given target set is considered. Sufficient conditions for a successful encounter of the initial system with the target set are given. The differential game problem of encounter between two objects with restricted manoeuvrability is solved. The equations of motion of the objects and the constraints imposed on the controls are given using the same relations as those in the differential game known /1/ as the "two-car game". It is assumed that both objects have restricted manoeuvrability and the game takes place over a fixed time interval. A positional strategy is constructed for the pursuer guaranteeing that it arrives at a previously specified distance from the pursued at the instant of game termination. The paper is related to work done in /1-14/.

1. Let us consider a conflict-controlled system described by the following differential vector equation:

$$x^{\cdot} = f(t, x, u, v),\ u \in P,\ v \in Q \tag{1.1}$$

Here x is the n-dimensional phase vector of the system, u and v are the r-dimensional vector controls subordinate to the first and second player respectively, P and Q are closed and bounded sets. The function $f(t, x, u, v)$ is assumed to be continuous and satisfying the Lipschitz conditions in x in every bounded region of the space $\{x\}$.

We define, in the space $\{x\}$, the convex closed set M as the target of the first player. The initial position of the game $\{t_0, x_0\}$ is fixed.

The admissible strategy of the first player U_ϑ is defined as a function which places, in correspondence with every position $\{t, x\}$, a closed set $U(t, x) \in P$, and the mapping $(t, x) \to U(t, x)$ semicontinuous from above, is multivalued with respect to the inclusion. The set of motions generated by the strategy U_ϑ and emerging from the point $\{t_*, x_*\}$ $(x_* = x(t_*))$, is denoted by $X(t_*, x_*, U_\vartheta)$. We will call the motion $x[t] = x[t, t_*, x_*, U_\vartheta]$ any absolutely continuous function satisfying the conditions $x[t_*] = x_*$, $x^{\cdot}[t] = F_U(t, x[t])$ for almost all $t \geqslant t_*$. Here $F_U(t, x) = \operatorname{co}\{f(t, x, u, v) \mid u \in U(t, x),\ v \in Q\}$.

The strategy U_ϑ guarantees, by definition, the encounter of the point $x(t)$ with the target M from the position $\{t_0, x_0\}$, at the instant $\vartheta \geqslant t_0$, provided that $x(\vartheta) \in M$ for any motion $x(\cdot) \in X(t_*, x_*, U_\vartheta)$.

We require to construct an admissible strategy U_ϑ which will guarantee that $x[t]$ will find itself in some specified neighbourhood of the set M at the instant $t = \vartheta$ from the position $\{t_0, x_0\}$.

Let us consider the following auxiliary system described by the linear differential equation

$$x^{\cdot} = A(t)x + u_1 - v_1,\ u_1 \in P_1,\ v_1 \in Q_1 \tag{1.2}$$

Here x is the n-dimensional phase vector of the system, u_1, v_1 are the r-dimensional vector controls subordinated to the pursuer and pursued respectively, P_1 and Q_1 are closed and bounded sets, and the matrix function $A(t)$ is assumed to be continuous.

Let a u-stable function $\varepsilon(t, x)$ be given for the auxiliary system (1.2), satisfying the condition

$$\varepsilon(t, x) \leqslant \varepsilon^0 \quad (\varepsilon^0 = \varepsilon(t_0, x_0) \geqslant 0) \tag{1.3}$$

Let R^n be an n-dimensional Euclidean space. We shall use the symbols $\langle x, y \rangle$ and $\|x\|$ to denote the scalar product of the vectors $x, y \in R^n$ and the norm of the vector x.

We will assume that the initial (1.1) and auxiliary (1.2) system are connected by the relation

$$\min_{u_1 \in P_1} \max_{v_1 \in Q_1} \langle s, A(t)x + u_1 - v_1 \rangle + c \geqslant \min_{u \in P} \max_{v \in Q} \langle s, f(t, x, u, v) \rangle \tag{1.4}$$

Here $c > 0$, s is an arbitrary n-dimensional vector and $\|s\| \leqslant \lambda$ where λ is the Lipschitz

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constant of the function $\varepsilon(t, x)$ with respect to the variable x, i.e.

$$\lambda = \sup |\varepsilon(t, x^{(1)}) - \varepsilon(t, x^{(2)})| \quad \|x^{(1)} - x^{(2)}\|^{-1}$$
$$(t, x^{(i)}) \in D, \ x^{(1)} \neq x^{(2)}$$

Here D is an $(n+1)$-dimensional region in the position space $\{t, x\}$, in which the game (1.1) is studied.

Let $D^*\varepsilon(t, x) | (s)$ be a conjugate derivative of the function $\varepsilon(t, x)$. Since the function $\varepsilon(t, x)$ is u-stable for system (1.2), therefore according to /5/ we have, for any $s \in R^n$,

$$D^*\varepsilon(t, x) | (s) \geqslant H_2(t, x, s) = \min_{u_1 \in P_1} \max_{v_1 \in Q_1} \langle s, A(t)x + u_1 - v_1 \rangle \tag{1.5}$$

We note that from the definition of a conjugate derivative it follows that

$$D^*\varepsilon(t, x) | (s) = \infty \quad (\|s\| > \lambda) \tag{1.6}$$

Let us write

$$\varepsilon^*(t, x) = \varepsilon(t, x) + c \cdot (\vartheta - t) \tag{1.7}$$

The function ε^* is u-stable for system (1.1).

Since

$$D^*\varepsilon^*(t, x) | (s) = D^*\varepsilon(t, x) + c \tag{1.8}$$

then to prove the u-stability of the function $\varepsilon^*(t, x)$ we must, according to /5/, prove the inequality

$$D^*\varepsilon^*(t, x) | (s) \geqslant H_1(t, x, s) = \min_{u \in P} \max_{v \in Q} \langle s, f(t, x, u, v) \rangle \tag{1.9}$$

for any $s \in R^n$.

Indeed, in $\|s\| \leqslant \lambda$, then from (1.6) and (1.8) we have $D^*\varepsilon^*(t, x) | (s) = \infty$, therefore (1.9) holds.

If on the other hand $\|s\| \leqslant \lambda$, then from (1.9) there follow (1.4), (1.5) and (1.8).

Let us denote by $W(t, x)$ the set of points $\{t, x\}$ satisfying the condition

$$\varepsilon^*(t, x) \leqslant \varepsilon^0 + c \cdot (\vartheta - t) \quad (\varepsilon^0 = \mathrm{const} > 0) \tag{1.10}$$

and relation

$$\varepsilon(\vartheta, x) = \rho(x, M) \quad (\rho(x, M) = \min_y \|x - y\|, \ y \in M) \tag{1.11}$$

We know /2/ that the set W is a u-stable bridge in the problem of the encounter of the system (1.1) with the $(\varepsilon^0 + c \cdot (\vartheta - t_0))$-neighbourhood of the set M.

If the initial position $\{t_0, x_0\}$ is such that $\varepsilon(t_0, x_0) = \varepsilon^0$, i.e. $\varepsilon^*(t_0, x_0) = \varepsilon^0 + c \cdot (\vartheta - t_0)$, then according to /2/ the strategy $U_e^{(\vartheta)}$ extremal with respect to the u-stable bridge $W(t, x)$, guarantees that $\varepsilon^*(\vartheta, x(\vartheta)) \leqslant \varepsilon^0 + c \cdot (\vartheta - t_0)$, i.e.

$$\varepsilon^*(\vartheta, x(\vartheta)) = \rho(x(\vartheta), M) \leqslant \varepsilon^0 + c \cdot (\vartheta - t_0)$$

Thus we have the following

Theorem. Let a certain u-stable function $\varepsilon(t, x)$ be given in the game of encounter of the auxiliary (linear) system (1.2) with the set M, satisfying the condition $\varepsilon(t, x) \leqslant \varepsilon^0$ where $\varepsilon^0 = \varepsilon(t_0, x_0)$, and let relation (1.5) connecting the initial (1.1) and auxiliary system (1.2), hold.

Then we can construct for the initial (non-linear) system (1.1) a strategy $U_e^{(\vartheta)}(t, x)$ extremal with respect to the u-stable bridge $W(t, x)$, guaranteeing the encounter of the system (1.1) from the initial position $\{t_0, x_0\}$, with the $(\varepsilon^0 + c \cdot (\vartheta - t_0))$-neighbourhood of the set M at the instant $t = \vartheta$,

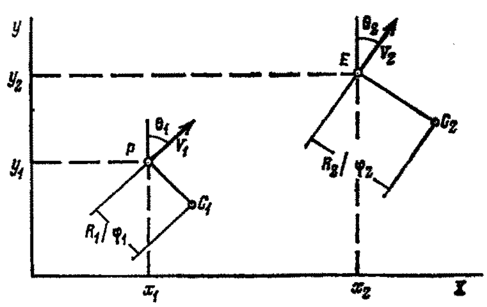

Fig.1

2. Let us consider a differential game of encounter between two objects. The equations of motion of the objects and the constraints imposed on the controls are given by the same relations as those used in the game problem known /1/ as the "two-car game".

The object P (pursuer) and E (evader) move with constant velocity in the XY plane (Fig.1). the motion of the object P is described by the equations /1/ ($i = 1$ corresponds to the object, P, and $i = 2$ to Q

$$x_i^{\cdot} = V_i \sin\theta_i,\ y_i^{\cdot} = V_i \cos\theta_i,\ \theta_i^{\cdot} = (V_i/R_i)\varphi_i,\ |\varphi_i| \leqslant 1 \tag{2.1}$$

Here V_1 and V_2 are constant velocities of P and E, R_1 and R_2 are the minimum radii of their trajectories, θ_1 and θ_2 are the angles between the OY axis and the vectors V_1 and V_2 respectively, and φ_1 and φ_2 are the scalar controls of P and E respectively. The points C_1 and C_2 denote the centres of curvature of the trajectories of P and E respectively (Fig.1).

The problem is considered in the time interval $[t_0, \vartheta]$. The payoff function is the distance between the objects P and E and the instant $t = \vartheta$ of game termination

$$\varepsilon(\vartheta) = \{[x_2(\vartheta) - x_1(\vartheta)]^2 + [y_2(\vartheta) - y_1(\vartheta)]^2\}^{1/2} \tag{2.2}$$

The pursuer P minimizes the game payoff (2.1)-(2.2) and the evader maximizes it.

We shall regard as the solution of the problem (2.1)-(2.2) the construction of a positional strategy $U_e^{(\vartheta)}(t, x, y)$ of the pursuer P guaranteeing that it will arrive at the instant $t = \vartheta$ at the previously known distance $\varepsilon^* \geqslant 0$ from the evader, i.e. $\varepsilon(\vartheta) \leqslant \varepsilon^*$.

3. We will consider the initial problem for the case (2.1)-(2.2), when the following expansions hold for θ_i ($\delta\theta_i$ is a fairly small quantity):

$$\theta_i = \theta_i^0 + \delta\theta_i \tag{3.1}$$

The auxiliary differential game of encounter will be linear. The motion of the objects P and E in the auxiliary problem will proceed according to the linear equation ($i = 1$ corresponds to the object P, and $i = 2$ to the object E)

$$x_i^{\cdot} = V_i(\sin\theta_i^0 + \delta\theta_i^0),\ y_i^{\cdot} = V_i(\cos\theta_i^0 - \delta\theta_i \sin\theta_i^0) \tag{3.2}$$
$$(\delta\theta_i)^{\cdot} = (V_i/R_i)\varphi_i,\ |\varphi_i| \leqslant 1 \quad (i = 1, 2)$$

Let us formulate the auxiliary game problem. Let the motion of the object P and the choice of control be given by (3.2). We will assume that the motion of the object E and the constraints imposed on its control are given by relations (2.1). The game is played in a fixed time interval $[t_0, \vartheta]$.

We will assume that the following inequality holds for the auxiliary game (3.2):

$$V_1^2/R_1 \geqslant V_2^2/R_2 \tag{3.3}$$

The payoff functional is given for the auxiliary game (3.2)-(3.3) by the same relation (2.2) as for the initial game (2.1). The pursuer P minimizes, and the evader E maximizes the value of $\varepsilon(\vartheta)$. The initial positions of the objects P and E in the auxiliary game (3.2)-(3.3) are given by the vectors $\{x_1^0, y_1^0, (\delta\theta)^0 = 0\}$ and $\{x_2^0, y_2^0, (\delta\psi)^0 = 0\}$ respectively, and the values of the geometrical coordinates coincide with the initial values of the geometrical coordinates of the objects P and E in the initial game (2.1)-(2.2).

4. Let us construct in the XY-plane the domains of accessibility up to the instant $t = \vartheta$, by the objects P and E, in the auxiliary game (3.2), corresponding to the initial positions $\{t_0, x_1^0, y_1^0, (\delta\theta_1)^0 = 0\}$ and $\{t_0, x_2^0, y_2^0, (\delta\theta_2)^0 = 0\}$.

We can see that the domains of accessibility $G^{(1)} = G^{(1)}(t, \vartheta)$ and $G^{(2)} = G^{(2)}(t, \vartheta)$ by the objects P and E in the auxiliary game (3.2)-(2.2) will be the segments of the straight lines A_1B_1 and A_2B_2 (Fig.2) perpendicular to the vectors $\bar{n}_1$ and $\bar{n}_2$ respectively, where

$$\bar{n}_i = \{V_i(\vartheta - t)\sin\theta_i^0;\ V_i(\vartheta - t)\cos\theta_i^0\}\ (i = 1, 2)$$

The point $A_1(x_1^+(\vartheta), y_1^+(\vartheta))$ (point $A_2(x_2^+(\vartheta), y_2^+(\vartheta))$ of the domain of accessibility $G^{(1)}$ ($G^{(2)}$) can be reached by the object P (object E) at the instant $t = \vartheta$, provided that the latter exercises constant control $\varphi_1 = +1$ ($\varphi_2 = +1$) over the whole time interval $t_0 \leqslant t \leqslant \vartheta$.

Similarly, the point $B_1(x_1^-(\vartheta), y_1^-(\vartheta))$ (point $B_2(x_2^-(\vartheta), y_2^-(\vartheta))$ of the domain of accessibility $G^{(1)}$ ($G^{(2)}$) can be reached by the object P (object E) at the instant $t = \vartheta$ provided that the latter exercizes constant control $\varphi_1 = -1$ ($\varphi_2 = -1$) over the whole time interval $t_0 \leqslant t \leqslant \vartheta$.

The object P (object E) reaches at the instant $t = \vartheta$ the point $O_1(x_1^0(\vartheta), y_1^0(\vartheta))$, which is the centre of the segment A_1B_1 (point $O_2(x_2^0(\vartheta), y_2^0(\vartheta))$ which is the centre of the segment A_2B_2), provided that it exercises a constant control $\varphi_1 = 0$ ($\varphi_2 = 0$) over the period $t_0 \leqslant t \leqslant \vartheta$.

Let us introduce the notation

$$r_i(t, \vartheta) = |O_iA_i| = |O_iB_i| \quad (i = 1, 2)$$

Integrating the Eqs.(3.2) at $\varphi_1 = +1$ or $\varphi_1 = -1$ ($\varphi_2 = +1$ or $\varphi_2 = -1$), we can calculate

$$r_i(t, \vartheta) = (V_i(\vartheta - t))^2/(2R_i)\ (i = 1, 2) \tag{4.1}$$

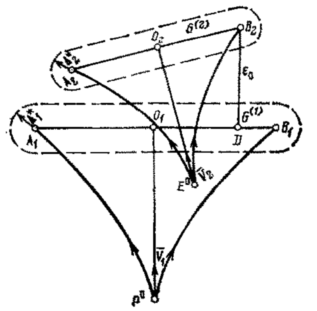

Fig.2

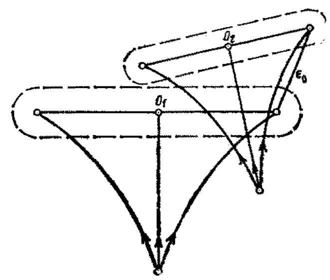

Fig.3

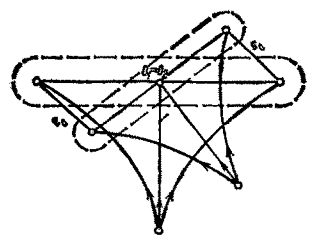

Fig.4

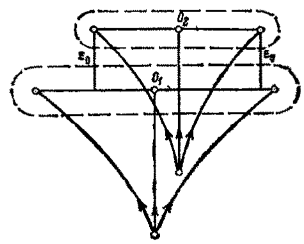

Fig.5

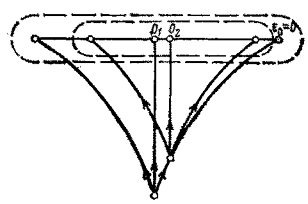

Fig.6

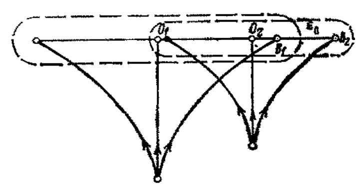

Fig.7

From (3.3) it follows that

$$r_1(t, \vartheta) > r_2(t, \vartheta), \ \forall t \in [t_0, \vartheta] \tag{4.2}$$

The centre of the domain of reachability $G^{(i)}$ in the game (3.2), (2.2) (the point O_i) has the following coordinates at the instant $t = \vartheta$:

$$\{x_{ic}, y_{ic}\} = \{x_i^0 + V_i(\vartheta - t)\sin\theta_i^0 + (\delta\theta_i)^0 \cos\theta_i^0 V_i(\vartheta - t);$$
$$y_i^0 + V_i(\vartheta - t)\cos\theta_i^0 - (\delta\theta_i)^0 \sin\theta_i^0 V_i(\vartheta - t)\} \ (i = 1, 2)$$

Thus we have constructed the domains of reachability by the objects P and E in the auxiliary linear game, which represent convex sets. Here, according to (4.2), the "radius" $r_1(t, \vartheta)$ of the domain of reachability $G^{(1)}$ by the object P is greater than the "radius" $r_2(t, \vartheta)$ of the domain of reachability of E.

5. Let us obtain an expression for the function of hypothetical desynchronization /2/ $\varepsilon_0(t, x_1, y_1, \delta\theta_1, x_2, y_2, \delta\theta_2)$ in the auxiliary game (3.2), (2.2). We consider once again the domains of reachability $G^{(1)}$ and $G^{(2)}$ (Figs.2-7).

The function of hypothetical desynchronization of the problem in question can be regarded as the distance between the domain of reachability $G^{(1)}$ (the segment of the line A_1B_1) and the most distant point of the domain $G^{(2)}$ (a point on the segment A_2B_2). All possible cases of the mutual distribution of the domains of reachability $G^{(1)}$ and $G^{(2)}$ are shown in Figs.2-7. Let us discuss them.

1°. The domains of reachability by the objects P and E are non-collinear and the domain $G^{(2)}$ can be fully projected onto $G^{(1)}$ (Fig.2). It is clear that in this case the most distant point of the domain $G^{(2)}$ will be one of the end points (in this case the point B_2). The point of $G^{(1)}$ nearest to it lies on the intersection of the perpendicular dropped from the end point of $G^{(2)}$ (point B_2) on the segment A_1B_1 (point D). Therefore in this case the function of hypothetical desynchronization has the meaning of the maximin of the distance between the points of $G^{(2)}$ and $G^{(1)}$ (the distance between the points B_2 and D). Thus the value of the function of hypothetical desynchronization is obtained here on the unique vector $\bar{l}$ ($\bar{l} = \overline{DB_2}$).

2°. The domains of reachability by P and E are also non-collinear, but unlike the previous case the domain $G^{(2)}$ cannot be fully projected on to the domain $G^{(1)}$. The farthest point of $G^{(2)}$ will be one of the end points (the point B_2), and the point of $G^{(1)}$ nearest to it will be one of the end points (the point B_1). Here the maximin of the distance between $G^{(2)}$ and $G^{(1)}$ is the distance between these two extreme points ($\bar{l} = \overline{B_1B_2}$).

3°. The domains of reachability $G^{(1)}$ and $G^{(2)}$ intersect, and their centres coincide, i.e. $O_1 = O_2$. In this case the maximin of the distance between the points of $G^{(2)}$ and $G^{(1)}$ is attained simultaneously at two pairs of points. The distance between the points A_2 and A_1 and between B_2 and B_1 is clearly $|A_2A_1| = |B_2B_1|$. Thus the value of the hypothetical desynchronization function (the maximin of the distance between $G^{(2)}$ and $G^{(1)}$ is attained simultaneously along the pair of vectors $\bar{l}$: $\bar{l}_1 = \overline{B_1B_2}$, $\bar{l}_2 = \overline{A_1A_2}$.

4°. The domains $G^{(1)}$ and $G^{(2)}$ are collinear (Fig.5). In this case the distance between any point of $G^{(2)}$ and the domain $G^{(1)}$ is the same. The value of the maximin of the distance between $G^{(2)}$ and $G^{(1)}$ is reached simultaneously on the infinite set of vectors $\bar{l}$.

5°. The domains $G^{(1)}$ and $G^{(2)}$ (Fig.6) lie on a single straight line and $G^{(2)} \subset G^{(1)}$. In this case the function of hypothetical desynchronization is clearly equal to zero.

6°. The domains $G^{(1)}$ and $G^{(2)}$ (Fig.7) lie on a single straight line, but $G^{(2)} \not\subset G^{(1)}$. Here the maximin of the distance between the domains $G^{(2)}$ and $G^{(1)}$ is equal to the distance between the end point of $G^{(2)}$ not belonging to $G^{(1)}$, and the end point of $G^{(1)}$ nearest to it. (In this case this is the distance between B_2 and B_1). The maximin value is attained on the unique vector $\bar{l}$ directed from the end of the domain $G^{(1)}$ to the end point of $G^{(2)}$. (In this case this is $\bar{l} = \overline{B_1B_2}$).

The above discussion implies that the domains of reachability $G^{(1)}$ and $G^{(2)}$ are convex, closed and bounded sets (segments of straight lines). According to (4.2) the "radius" of $G^{(1)}$ is greater than the "radius" of $G^{(2)}$, i.e. $r_1(t, \vartheta) > r_2(t, \vartheta)$ for all $t \in [t_0, \vartheta]$.

The hypothetical desynchronization function /2/ in the game (3.2), (2.2) has the form

$$\varepsilon_0 = \varepsilon_0(t, x_1, y_1, \delta\theta_1, x_2, y_2, \delta\theta_2) = \max_{\{x_2, y_2\} \in G^{(2)}} \min_{\{x_1, y_1\} \in G^{(1)}} \rho\{G^{(2)}(t; \vartheta); G^{(1)}(t, \vartheta)\} \tag{5.1}$$

Here ρ is the distance between the point $\{x_2, y_2\} \in G^{(2)}$ and $G^{(1)}$. Let the maximum of the right-hand side of (5.1) be attained at some point $\{x_2^*, y_2^*\} \in G^{(2)}$, and the minimum at some point $\{x_1^*, y_1^*\} \in G^{(1)}$.

We can now write the function ε_0 in the form

$$\begin{aligned}
&\varepsilon_0 = [q_1^2 + q_2^2]^{1/2} \\
&q_1 = q_1(t, x_1, x_2, \delta\theta_1 \ \ \delta\theta_2) = x_2^* - x_1^* = \\
&\quad (\vartheta - t)(V_2 \sin\theta_2^0 - V_1 \sin\theta_1^0) + (\vartheta - t)(V_2 \cos\theta_2^0\delta\theta_2^0 - \\
&\quad V_1 \cos\theta_1^0\delta\theta_1^0) + \\
&\quad ((\vartheta - t)^2/2)((V_2^2 \cos\theta_2^0\varphi_2^*)/R_2 - (V_1^2 \cos\theta_1^0\varphi_1^*)/R_1)) + x_2^0 - x_1^0 \\
&q_2 = q_2(t, y_1, y_2, \delta\theta_1, \delta\theta_2) = y_2^* - y_1^* = \\
&\quad (\vartheta - t)(V_2 \cos\theta_2^0 - V_1 \cos\theta_1^0) - (\vartheta - t)(V_2 \sin\theta_2^0\delta\theta_2^0 - \\
&\quad V_1 \sin\theta_1^0\delta\theta_1^0) - \\
&\quad ((\vartheta - t)^2/2)((V_2^2 \cos\theta_2^0\varphi_2^*)/R_2 - (V_1^2 \sin\theta_1^0\varphi_1^*/R_1)) + y_2^0 - y_1^0
\end{aligned} \tag{5.2}$$

Here $\varphi_1 = \varphi_1^*$, $\varphi_2 = \varphi_2^*$ denote the values of the controls of P and E respectively, satisfying the constraints (3.2) and ensuring the arrival of the objects P and E at the points

$$\{x_1^*(\vartheta), \ y_1^*(\vartheta)\} \in G^{(1)}, \ \{x_2^*(\vartheta), \ y_2^*(\vartheta)\} \in G^{(2)}$$

According to /2/ it can be shown that in the first, second and sixth case (Figs.2,3,7) of the auxiliary game of encounter (3.2), (2.2) we have the regular case of the game of encounter.

The positions of the game (3.2), (2.2) corresponding to the third, fourth and fifth case (Figs.4,5,6) form a regular set S. They are characterized by the fact that the value of the program maximin ε_0 is attained for them simultaneously on two or more vectors $\bar{l}$. It can be shown that for the positions of the game (3.2), (2.2) belonging to S, the function ε_0 will be φ-stable.

Below we will show that using the φ-stability of the function of program maximin ε_0 we can construct the required strategy $U_e^{(\vartheta)}(t, x, y)$ of the object P from the condition of maximum gradient of the function ε_0, where the gradient exists.

6. Let us write Eqs.(2.1) in the form of a 6-th order system, making the change of variables according to the formulas

$$z_1 = x_1, \ z_2 = y_1, \ z_3 = \theta_1, \ z_4 = x_2, \ z_5 = y_2, \ z_6 = \theta_2 \tag{6.1}$$

Now the initial conflict-controlled system will have the form

$$\begin{aligned}
&z_1^{\cdot} = V_1 \sin z_3, \ z_2^{\cdot} = V_1 \cos z_3, \ z_3^{\cdot} = (V_1/R_1)\varphi_1 \\
&z_4^{\cdot} = V_2 \sin z_6, \ z_5^{\cdot} = V_2 \cos z_6, \ z_6^{\cdot} = (V_2/R_2)\varphi_2 \\
&-1 \leqslant \varphi_1, \varphi_2 \leqslant 1
\end{aligned} \tag{6.2}$$

Here φ_1 and φ_2 are the controls of P and E respectively, and $z = \{z_1, z_2, \ldots, z_6\}$ is the vector of phase coordinates. We shall regard the vector $z(t) = \{t, z_1, z_2, \ldots, z_6\}$ as the position of the game, and the time of the game is fixed $[t_0, \vartheta]$.

In accordance with (2.2), the set of termination of the game (6.2) in question is given by the relation

$$M = \{t = \vartheta;\ z\colon [(z_4 - z_1)^2 + (z_5 - z_2)^2]^{1/2} = 0\} \tag{6.3}$$

The aim of the object P is to ensure that $z(\vartheta)$ reaches M by the time $t = \vartheta$, and the second object tries to prevent this.

The fixed initial position of the game (6.2), (6.3) $\{t_0, z_1^0, z_2^0, z_3^0 = \theta_1^0, z_4^0, z_5^0, z_6^0 = \theta_2^0\}$.

Let us now write the equations of the auxiliary system (3.2) in the form of a 6-th order system, making the change of variable according to the formula

$$z_1 = x_1,\ z_2 = y_1,\ z_3 = \delta\theta_1,\ z_4 = x_2,\ z_5 = y_2,\ z_6 = \delta\theta_2 \tag{6.4}$$

Now the equations of the auxiliary, conflict-controlled system will have the form

$$\begin{aligned}
&z_1^{\cdot} = V_1(\sin z_3^0 + z_3\cos z_3^0),\ z_2^{\cdot} = V_1(\cos z_3^0 - z_3\sin z_3^0)\\
&z_3^{\cdot} = (V_1/R_1)\varphi_1,\ z_4^{\cdot} = V_2(\sin z_6^0 + z_6\cos z_6^0)\\
&z_5^{\cdot} = V_2(\cos z_6^0 - z_6\sin z_6^0),\ z_6^{\cdot} = (V_2/R_2)\varphi_2\\
&-1 \leqslant \varphi_1,\ \varphi_2 \leqslant 1
\end{aligned} \tag{6.5}$$

Here φ_1 and φ_2 are the controls of P and E respectively.

Determination of the position and time of the game, the termination set M, the initial position and the aims of the objects P and E in the auxiliary game are all the same as in the initial game (6.2), (6.3).

Let us consider, for the positions $\{t, z\} \overline{\in} S$, the function of program maximin ε_0 in the auxiliary game (6.5), (6.3). We have, for the positions in question, a regular case of a game of encounter of the system (6.5) with M. We know /2/ that in the regular case of the game of encounter the function $\varepsilon_0(t, z_1, z_2, \ldots, z_6)$ is differentiable and satisfies the relation

$$\min_{\varphi_1}\max_{\varphi_2} d\varepsilon_0/dt = 0 \tag{6.6}$$

Taking into account (6.4), we have the following relations:

$$\begin{aligned}
&\varepsilon_0(t, z_1, z_2, \ldots, z_6) =\\
&\quad [q_1^2(t, z_1, z_3, z_4, z_6) + q_2^2(t, z_2, z_3, z_5)]^{1/2}\\
&q_1 = (\vartheta - t)(V_2\sin\theta_2^0 - V_1\sin\theta_1^0) + (\vartheta - t)(z_6V_2\cos\theta_2^0 -\\
&\quad z_3V_1\cos\theta_1^0) + ((\vartheta - t)^2/2)((V_2^2\cos\theta_2^0\varphi_2^*)/R_2 -\\
&\quad (V_1^2\cos\theta_1^0\varphi_1^*)/R_1) + z_4 - z_1;\\
&q_2 = (\vartheta - t)(V_2\cos\theta_2^0 - V_1\cos\theta_1^0) - (\vartheta - t)(z_6V_2\sin\theta_2^0 -\\
&\quad z_3V_1\sin\theta_1^0) - ((\vartheta - t)^2/2)((V_2\cos\theta_2^0\varphi_2^*)/R_2 -\\
&\quad (V_1^2\sin\theta_1^0\varphi_1^*)/R_1) + z_5 - z_2
\end{aligned} \tag{6.7}$$

Here φ_1^* and φ_2^* are the program controls of the objects P and E respectively, on which the value of the program maximin is reached in the course of computing the function ε_0. According to Sect.1 we have

$$\varepsilon^*(t, z) = \varepsilon_0(t, z) + c\cdot(\vartheta - t) \tag{6.8}$$

where the constant $c > 0$ will be determined below.

Let us denote by $s_i(t, z)$ the partial derivatives of the function ε^*, i.e. $s_i(t, z) = (\partial\varepsilon^*/\partial z_i)$ $(i = 1, 2, \ldots, 6)$. It can be confirmed that

$$s_1^2(t, z) + s_2^2(t, z) = 1,\ s_4^2(t, z) + s_5^2(t, z) = 1 \tag{6.9}$$

7. In accordance with Sect.1 and (6.7) φ-stable bridge $W(t, z)$ in the problem of the encounter of the initial system (6.2), (6.3) with the $(\varepsilon^0 + c\cdot(\vartheta - t_0))$-neighbourhood of the set M, consists of the positions satisfying the relation

$$\varepsilon^*(t, z) \leqslant \varepsilon^0 + c\cdot(\vartheta - t_0) \tag{7.1}$$

and the condition

$$\begin{aligned}
&\varepsilon^*(\vartheta, z(\vartheta)) = \rho(z(\vartheta), M) \leqslant \varepsilon^0 + c\cdot(\vartheta - t_0)\\
&\varepsilon^0 = \varepsilon_0(t_0, z_0)
\end{aligned} \tag{7.2}$$

Taking (6.8) into account, the strategy $U_e^{(\vartheta)}$, extremal with respect to the stable bridge $W(t, z)$ defined by the expressions (7.1), (7.2), will have, according to /3/, the following form:

When $\{t, z\} \overline{\in} S$:

$$\varphi_1 = \operatorname{sign}(-\cos\theta_1^0 s_1(t, z) + \sin\theta_1^0 s_2(t, z) \tag{7.3}$$

if

$$\sigma(t, z) = -\sin\theta_1^0 s_2(t, z) + \cos\theta_1^0 s_1(t, z) \neq 0$$

if

$$\sigma(t, z) = 0 : \varphi_1 \in [-1, 1]$$

When $\{t, z\} \in S : \varphi_1 \in [-1, 1]$.

The last two relations mean that the control φ_1 can take any value satisfying the constraint (6.2).

8. Let us consider the initial system (6.2) and auxiliary system (6.5). In order to compare these systems, we introduce the vector s, defining it as follows.

If $\{t, z\} \overline{\in} S$ and $\sigma(t, z) \neq 0$, then $s = \mathrm{grad}_z\, \varepsilon^*(t, z)$. If $\{t, z\} \overline{\in} S$ and $\sigma(t, z) = 0$, then $s = 0$. We write $s = 0$ for the positions $\{t, z\} \in S$.

Next we estimate the magnitude of the difference between the scalar product of the vector s and the right-hand side of the initial system (6.2), and the scalar product of the same vector s and the right-hand side of the auxiliary linear system (6.5).

Taking into account that the third and sixth equation of the system (6.2) and (6.5) are the same, and the components of the vector s have the form (6.9), we find that the estimate Δ^* of the difference between the scalar products mentioned above, taking (3.1) into account, will be

$$\Delta^* \leqslant {}^1/_2\, [V_1\, (\delta\theta_1^*)^2 + V_2\, (\delta\theta_2^*)^2]$$

Here the methods of averaging /15/ can be used to compute the quantities $\delta\theta_1^*$, $\delta\theta_2^*$.

Assuming that $c = \Delta^*$, we can easily confirm that the inequality (1.4) holds for the initial system (6.2) as well as for the auxiliary system (6.5).

Thus we see that all conditions of the theorem given above hold. This implies that the strategy $U_e^{(\vartheta)}$ defined by expressions (7.3) guarantees the encounter of the initial system (6.2) with the $(\varepsilon^0 + \Delta^*(\vartheta - t_0))$-neighbourhood of the set M at the time $t = \vartheta$.

Note 1. Let us return to the formulation of the auxiliary game of encounter given in Sect.3 of this paper. Consider the motions of the object P in the XY plane caused by the initial system (2.1) and auxiliary system (3.2). We will assume here that the initial position at the instant $t = t_0$ is the same and the object P uses at $t_0 \leqslant t \leqslant \vartheta$ the same control $\varphi_1 = \varphi_1^*$ in both cases, satisfying the constraint (2.1). It can be shown that the position of the object P moving according to Eq.(3.2), will differ at the instant $t = \vartheta$ with an accuracy of the order $o(\delta\theta_1)$, from the position of P moving in accordance with the initial Eq.(2.1) by an amount

$$\Delta_1^* = \Delta_1(\varphi_1^*) = ({}^1/_2)\, V_1\, (\delta\theta_1^*)^2\, (\vartheta - t)$$

The value of $\delta\theta_1^*$ is found as follows. First we integrate the third equation of system (3.2) at $t_0 \leqslant t \leqslant \vartheta$ for $\varphi_1 = \varphi_1^*$ and initial condition $\delta\theta_1(t_0) = 0$, and average the resulting value /15/.

Using exactly the same arguments we can obtain an estimate for the difference between the position of the object E moving according to the initial Eq.(2.1), and its position at $t = \vartheta$ when it moves according to the linear system (3.2), in the form

$$\Delta_2^* = \Delta_2(\varphi_2^*) = ({}^1/_2)\, V_2\, (\delta\theta_2^*)^2\, (\vartheta - t)$$

Here $\varphi_2 = \varphi_2^*$ is the control of E. The value of $\delta\theta_2^*$ is found exactly as before.

Using the above arguments, we arrive at the following qualitative conclusion. Let the object $P(E)$ moving according to the auxiliary system (3.2) from a fixed initial position at the instant $t = t_0$, with some control $\varphi_1 = \varphi_1^*\ (\varphi_2 = \varphi_2^*)$, arrive at the instant $t = \vartheta$ at some point $A_1(A_2)$ of its domain of reachability $G^{(1)}(G^{(2)})$. Then from what was said before it follows that if the object $P(E)$ moves in accordance with the initial system (2.1) from the same initial position at $t = t_0$ and with the same control $\varphi_1 = \varphi_1^*\ (\varphi_2 = \varphi_2^*)$, then it will appear at the instant $t = \vartheta$ at some point on a circle with its centre at the point $A_1(A_2)$, of radius $r_1^*(r_2^*)$ where $r_1^* = \Delta_1^*\ (r_2^* = \Delta_2^*)$. From this it follows that the domain of accessibility of the initial system (2.1) will be situated in some ε^*-neighbourhood of the auxiliary system (3.2). Clearly, we have $\varepsilon^* = \Delta_1^*$ for (3.2) and $\varepsilon^* = \Delta_2^*$ for the system (3.2) with $i = 2$ (Figs.2-7).

The last assertion implies that having the above estimates for the initial game problem available, we could have restricted our investigation to considering the linear (auxiliary) differential game of encounter only.

Note 2. All the arguments used in this paper can be extended to the case when the velocities of the objects P and E are known functions of time.

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ON AN ALTERNATIVE FOR PURSUIT-EVASION GAMES IN AN INFINITE TIME INTERVAL*

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The structure of phase space of differential pursuit-evasion games is studied for the case when the evader is subjected to information discrimination with an advance by $\delta > 0$, $\delta = \text{const}$. The method of transfinite iteration of the Pshenichnyi operator is used to establish an alternative for differential pursuit-evasion games in an infinite time interval.

1. Definition of the basic concepts.

Consider a differential pursuit-evasion game in the phase space $\mathbf{R}^d$ with the equation of motion

$$z^{\cdot} = f(z, u, v),\ u \in P \subset \mathbf{R}^p,\ v \in Q \subset \mathbf{R}^q \tag{1.1}$$

and the terminal set $M \subset \mathbf{R}^d$. The aim of the pursuer who controls the parameter u is to bring the phase point from its initial position to the set M; the aim of the evader who controls the parameter v, is the opposite. At every instant of time $t \geqslant 0$ the pursuer has access to up to date information $z(t)$ and $v(s), t \leqslant s \leqslant t+\delta$ (everywhere in $\delta > 0$). Following /1/, we assume that the number δ is chosen by the evader at the start of the motion, and does not change during the motion (comp. with /2/).

The type of informability described above is realized below by separating specific classes of the strategies $\mathbf{P}, \mathbf{Q}$ of the pursuer and evader respectively (for brevity we will call such a pair of strategies the game). Here every triad $\xi \in \mathbf{R}^d, U \in \mathbf{P}, V \in \mathbf{Q}$ generates a unique trajectory $z(t; \xi, U, V)$, $t \geqslant 0$.

Let I denote either a segment of the form $[0, \tau]$, or a semi-axis $[0, \infty)$. By definition, a pursuit originating at the initial point ξ can be successfully completed in the interval I, if a strategy $U \in \mathbf{P}$ exists such that an inclusion $z(t; \xi, U, V) \in M$ occurs at some $t \in I$ whatever $V \in \mathbf{Q}$ is. Similarly, escape from the point ξ is possible in the interval I, if a strategy $V \in \mathbf{Q}$ exists such that we have $z(t; \xi, U, V) \overline{\in} M$ for any $t \in I$, whatever $U \in \mathbf{P}$ is.

Let I^+ (or I^-) denote the set of all points from which the pursuit (or possibly escape)

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